GEOMETRICAL ASPECTS OF TURBULENCE: EXTREME POINTS, DISSIPATION ELEMENTS AND STREAMLINES

P. SCHAEFER[1,a], M. GAMPERT[1], N. PETERS[1]

[1] Institute for Combustion Technology, RWTH Aachen, 52056, Germany

[a]Corresponding author: Tel.: +49/241/8094619; Fax: +49/241/8092923; Email: pschaefer@itv.rwth-aachen.de

KEYWORDS:

Main subjects: Streamlines, isosurface, flow visualization
Fluid: Homogeneous Isotropic Turbulence
Visualization method(s): Computational
Other keywords: Streamline segments, dissipation elements, stagnation points

ABSTRACT: New geometrical aspects of homogeneous isotropic turbulent flow fields obtained from direct numerical simulations (DNS) are visualized and analyzed based on the definition of different flow inherent geometries. The latter are dissipation elements based on the instantaneous turbulent kinetic energy field and streamline segments. Dissipation elements are intimately related to local extreme points of the underlying scalar field which are connected by gradient trajectories. Streamlines follow the local frozen velocity field and are partitioned into segments based on local extrema of the absolute value of the velocity along the streamline. These local extrema define an isosurface in space which also contains the local and global extrema of the instantaneous turbulent kinetic energy field and thus the endingpoints of dissipation elements. This isosurface is analyzed in various regards with respect to the geometry of dissipation elements as well as streamlines and streamline segments. The topological behavior of the isosurface is theoretically explained in the vicinity of stagnation points which, being absolute minima of the turbulent kinetic energy field, are also situated in the isosurface. An expansion of the isosurface shows that locally, indifferent of the type of stagnation point, it is a degenerated cone-type surface where two folds touch.

INTRODUCTION: In the course of turbulence research a manifold of different strategies has been devised to explain the complex motion of a turbulent fluid flow. The spectrum of approaches reaches from purely mathematical theories based solely on the Navier-Stokes equations, such as two-point correlations which have yielded the famous and celebrated 4/5th law [1], to dimensional arguments which has led to the prediction of the -5/3rd scaling of the energy spectrum in the inertial subrange [2]. Another approach which has greatly benefited from the recent advances in high performance computing is to analyze turbulent inherent geometries. In 1971, Corrsin [3] asked the question: "*What types (of geometry) are naturally identifiable in turbulent flows*?". In this spirit, vortex structures have been identified and analyzed for instance by She et al. [4] and Kaneda et al. [5]. They were found to form tubes in regions of high vorticity, while a sheet-like structure was identified in regions of low vorticity. However, vortex tubes and sheets do not allow a unique and space-filling decomposition of the flow field into unambiguous sub-ensembles. This problem was overcome by Wang and Peters [6, 7] in their concept of dissipation elements, an approach which has its roots in early works by Gibson [8] who analyzed the role of extreme points in turbulent scalar mixing processes. This concept, based on gradient trajectories, allows the decomposition of turbulent scalar fields into smaller sub-units. By calculating gradient trajectories in direction of ascending and descending scalar gradients, a local minimum and a local maximum point are reached. Dissipation elements are then defined as the spatial region from which all gradient trajectories reach the same pair of maximum and minimum points in a scalar field. They may then be parameterized by the linear distance between and the scalar difference at the extreme points which makes them amenable to statistical analysis. The most important feature of dissipation elements is that they are space-filling and unambiguous, meaning that at any instant in time the turbulent scalar field can be decomposed in a determined manner. Then, based on the much simpler conditional statistics within the dissipation elements and the knowledge of their statistical distribution (in terms of joint probability density functions) the complicated statistics of the entire scalar field can be reconstructed. Successful examples of this approach can be found in [7] and [9].

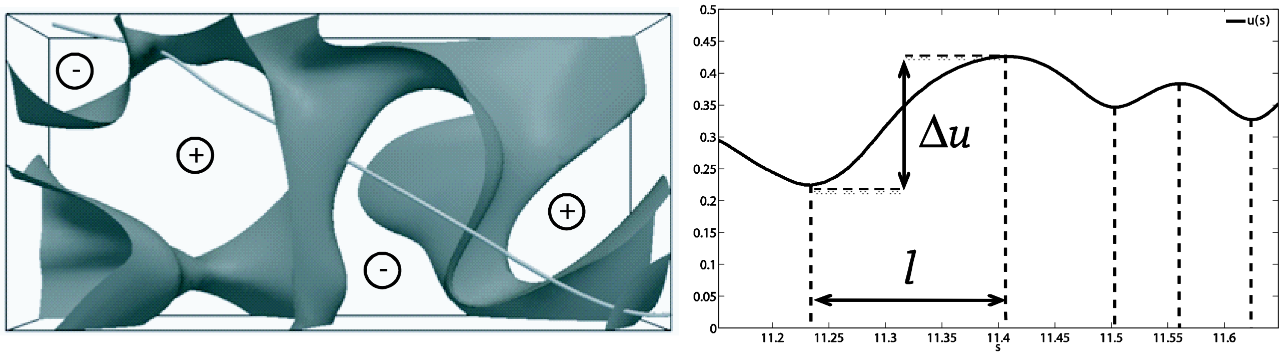
One major shortcoming of the theory of dissipation elements is that it is only applicable to turbulent scalar fields. In order to also apply this successful theory to the turbulent velocity field itself, Wang [10] proposed to study streamlines in turbulent velocity fields. The geometrical properties of particle paths (the analogon to streamlines in an evolving turbulent field) have for instance been studied by Rao [11], Braun et al. [12] and Scagliarini [13], whose ideas have been extended to the geometrical properties of streamlines by Schaefer et al. [14]. Streamlines are not Galilei invariant, meaning that the chosen frame of reference determines the streamline topology. Thus, one has to choose an appropriate frame of reference when analyzing turbulent flow fields based on streamlines. In the course of this work this frame of reference will be the fluctuating velocity field with zero mean for two reasons: first, from a geometrical point of view we are only interested in the geometry and topology of the fluctuating field, which is often used to isolate "pure" turbulent physics without the interaction with solid walls, mean gradients or alike. Second, it has been shown that there exists a frame, in which the so called streamline persistence is maximized [15]. Streamlines are considered persistent if their geometry changes slowly enough for a particle to approximately follow their path for a significantly long time. In that case, particles initially close to each other will only separate once they approach a straining stagnation point, where streamlines diverge. For isotropic turbulence, the case considered in this work, it could be shown that the appropriate frame of reference is the one where all mean velocity components vanish, i.e. the fluctuating velocity field [16]. In chapter 2 the mathematical basis of the isosurface is laid out and based on DNS data geometrical relations of streamlines, extreme points and dissipation elements are visualized together with the isosurface. In chapter 3 the focus is laid upon the local behavior of the isosurface in the vicinity of stagnation points and its local topology is shown and visualized. Concluding remarks are given in the last chapter.

THE du/ds=0 ISOSURFACE AND ITS CHARACTERISTICS: In an evolving turbulent velocity field $u_i(x_i, t)$, streamlines can be traced at any instant t in the frozen field from any point in space by following the normalized tangent vector $t_i = u_i/u$, where $u = (u_i\, u_i)^{1/2}$ denotes the absolute value of the velocity vector. Streamlines are, different from gradient trajectories in a scalar field, a-priori infinitely long, unless they hit a stagnation point, in which all three velocity components vanish and they diverge. Wang [10] has proposed to divide streamlines into segments based on local extreme points of u along the streamline. Segments are then bound by two extrema, i.e. points where the velocity gradient in streamline direction,

$$u_s \equiv \frac{\partial u}{\partial s} = t_i \frac{\partial u}{\partial x_i}, \qquad (1)$$

vanishes. From the above definition it follows readily that streamline segments end and begin where $u_s \equiv \partial u/\partial s = 0$. Based on eq. (1) we see that this condition defines an isosurface of the scalar field u_s which possess some remarkable features to be explored in the current work.

The isosurface divides space into two regions, namely a region, in which $u_s > 0$ containing all positive segments to be denoted with (+) and a region within which $u_s < 0$, containing all negative segments to be denoted with (-). Thus, streamlines starting from an arbitrary point in space will intersect the isosurface thereby entering alternatingly into regions denoted with (+) and (-) signs. The situation is illustrated in Figure 1(a) where the isosurface as well as a streamline is shown for a turbulent flow field calculated by DNS in three-dimensional space.



(a) (b)

Figure 1: (a) Isosurface defined by $u_s = 0$ and streamline. (b) Variation of u along the streamline over the arclength (arbitrary units) with segments.

The streamline enters the box from the top left corner and intersects the corrugated isosurface five times before leaving the box at the lower right corner. The five intersections define the boundaries of four segments along the streamline. Figure 1(b) shows the corresponding variation of $\mathbf{u}$ along the same streamline in a one-dimensional plot, where the four segments are demarcated with dotted lines whose locations correspond to the intersections shown in Figure 1(a). The variation is plotted over the arclength $\mathbf{s}$ so that the horizontal distance between two dotted lines corresponds to the length **l** of the segment, while the velocity difference at the ending points, i.e. the difference at the points where the streamline intersects the isosurface is labeled $\mathbf{\Delta u}$.

Figure 2 shows a bundle of streamlines passing alternatingly through positive and negative regions of space. It is well visible that in the positive region (red part of the streamlines) the streamline bundle contracts due to mass conservation, while in the negative region (blue part of the streamlines) the bundle diverges. In fact, it has been shown in [10] that the Gaussian curvature of streamlines, which corresponds to the divergence of the unit tangent vector ti, $\boldsymbol{\kappa_G = \partial t_i / \partial x_i}$ can be expressed by virtue of the continuity equation as $\boldsymbol{\kappa_G = -u_s / u}$ yielding a divergence of bundles in negative regions and a contraction in positive ones. In addition, this relation shows that the Gaussian curvature vanishes on the isosurface so that locally neighboring streamlines are parallel when passing through the isosurface. In that sense the isosurface could also be called "zero divergence surface of streamlines".

Apart from these properties, which are particularly important for streamline-based analyses, it follows readily that all local extreme points (of the field of the absolute value of the velocity u and the turbulent kinetic energy $k = u^2/2$, i.e. points where $\nabla u = \nabla k = 0$, lie in the isosurface, as

$$\nabla k = u \nabla u = 0 \Leftrightarrow \vec{t} \nabla u = 0 \qquad (2)$$

Figure 3(a) shows the distribution of extreme points of the instantaneous kinetic energy field in space where red dots mark local maximum points and blue dots local minimum points. Green dots indicate stagnations points, i.e. critical points of the velocity field where locally all velocity components vanish simultaneously $u_i = 0$. It is obvious that as $u \geq 0$ stagnation points form a sub-group of local minimum points, namely absolute minima of the turbulent kinetic energy. The local topology of the isosurface in the vicinity of the stagnation point deserves special attention and will be subject of the following chapter.

The isosurface itself can further be subdivided into two (not necessarily simply connected; isolated islands can exist) parts, one of which contains all minimum points (minimal surface), while the other contains all maximum points (maximal surface). The demarcation line between these two regions is the ensemble of points where streamlines are tangent to the isosurface. Formally, this amounts to the condition $t_i n_i = 0$, where n_i denotes the unit normal vector to the surface.

Figure 4 shows such a partitioning of the surface into minimal (light grey) containing all local minimum points and a maximal region (dark grey) containing all local maximum points.

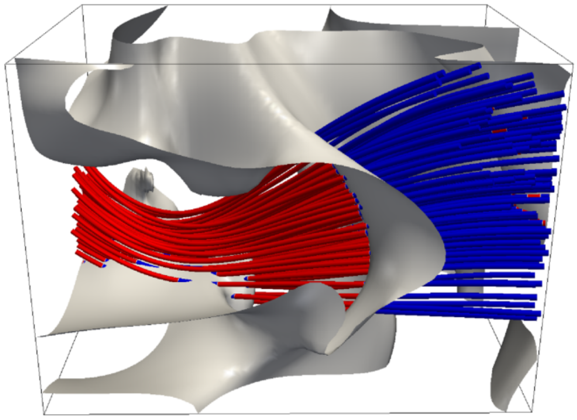


Figure 2: Bundle of streamlines passing alternatingly through positive (red) and negative (blue) regions of space.

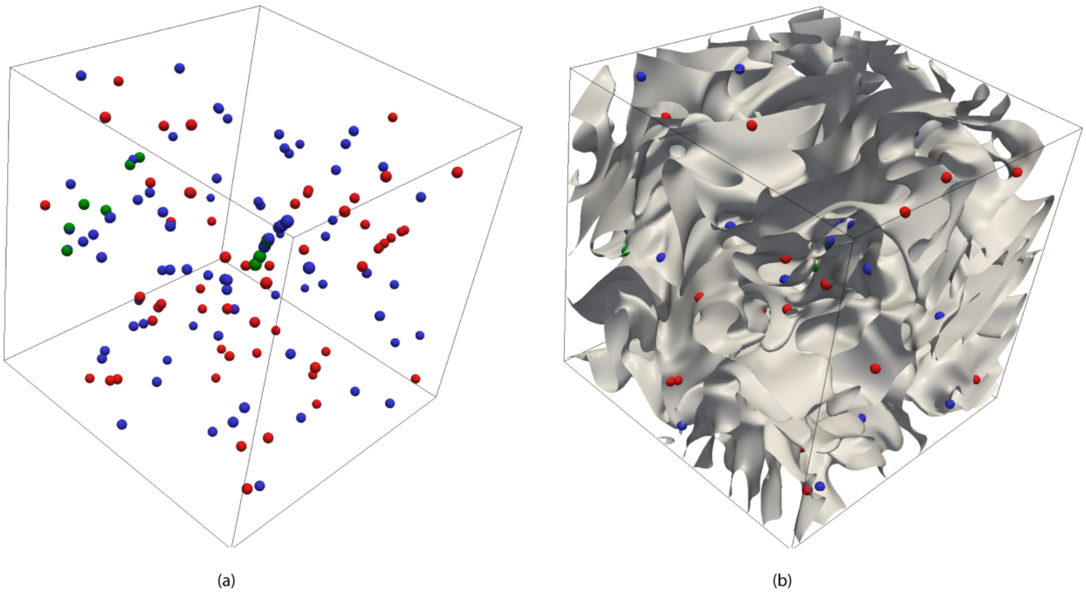


Figure 3: (a) Local extreme points of the k-field. (b) $u_s = 0$ isosurface containing all extreme points. Blue dots: minimum points, red dots: maximum points, green dots: stagnation points.

From the above analysis it follows that the theory of dissipation elements (based on the instantaneous u- or k-field, respectively), is closely connected to the isosurface and thus to the theory of streamline segments, as dissipation elements start and end in local extreme points which all lie in the isosurface. Figure 5 shows two examples of such elements and their gradient trajectories based on the instantaneous k-field. As is obvious, dissipation elements are corrugated and intertwisted structures which can intersect one or more folds of the surface.

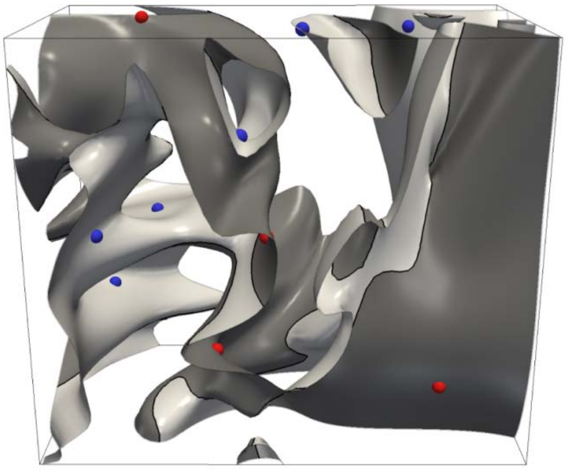


Figure 4: Subdivision of the $u_s = 0$ isosurface into minimal (light grey) and maximal (dark grey) surface regions with demarcation line (black). Minimum points (blue dots) of the k-field lie in the light grey region, maximum points (red dots) in the dark grey region of the surface.

However, they always (as they begin in a minimum and end in a maximum) connect a minimal with a maximal surface. Nonetheless, the fact that the ending points must at any instant lie in the isosurface has the implication that the dynamics of extreme points of the field are intimately related to those of streamline segments. The intimate relationship between the dynamics governing local extreme points is especially important for small dissipation elements. These connect two adjacent extreme points of opposite sign and will drift towards and annhilate each other once they join [6]. As both extreme points have to stay on the isosurface during this process but on two different sides of the demarcation line between minimal and maximal surface regions, the two will approach the demarcation line until they annihilate each other. In that sense the demarcation line plays the role of a sink for small dissipation elements. On the other hand, the gradient trajectories filling the space of larger elements will be likely to intersect the isosurface, as can be observed for the left dissipation element for example in figure 5 close to the minimum point.

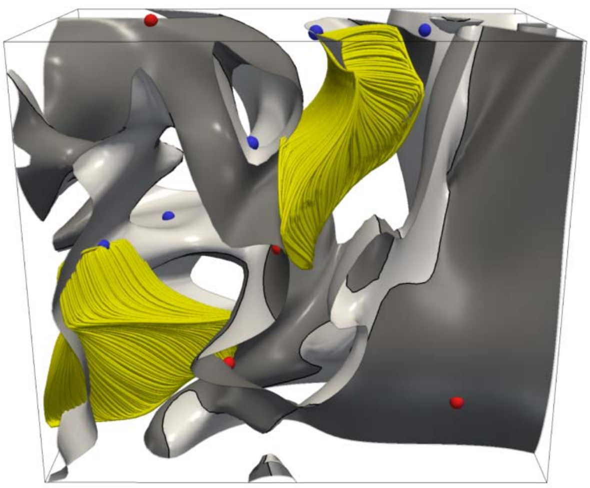


Figure 5: Two examples of dissipation elements embedded in the $u_s = 0$ isosurface with local extrema.

Let us denote with $b_i = (\partial k/\partial x_i)/|\nabla k| = (\partial u/\partial x_i)/|\nabla u|$ the unit tangent vector of gradient trajectories in the turbulent kinetic energy field (which are identical with those of the u field). It then follows that $cos\,\phi = b_i t_i$, the cosine of the angle between streamlines and gradient trajectories is obtained from

$$u\partial u/\partial s = \cos\phi\ \partial k/\partial b \qquad (3)$$
$$\partial u/\partial s = \cos\phi\ \partial u/\partial b$$

where the latter derivatives are the gradient in gradient trajectory direction, thus $\partial/\partial b = b_i\partial/\partial x_i$. From eq. (3) we conclude that streamlines and gradient trajectories must be perpendicular to each other when intersecting the isosurface, as $\partial u/\partial s = 0$ while $\partial k/\partial b \neq 0$ so that $\cos\phi \equiv 0$ which means that locally on the isosurface the instantaneous net convective transport of the turbulent kinetic energy is perpendicular to its diffusive transport.

BASIC PROPERTIES OF STRAGNATION POINTS: In the following we are interested in the local geometry of the isosurface in the vicinity of a stagnation point. Being a local minimum such a point should lie in the isosurface, however as it is a critical point of the flow field geometrical features in its vicinity are of particular interest and not a-priori clear. Stagnation points have mostly been studied in the context of flow visualization and to determine the local topology of the turbulent flow field. In [18] it has been shown based on previous works [18] that there exist four different types of stagnation points in incompressible flows (two more types can be identified in compressible ones). Without loss of generality let us shift the origin of a local cartesian coordinate system in the stagnation point. Then, a linearization of the flow field around the stagnation point yields

$$u_i \approx A_{ij}\, x_j + O(|x|^2), \qquad (4)$$

where $A_{ij} = \partial u_i/\partial x_j$ denotes the velocity gradient tensor at the stagnation point. The latter can be decomposed as $A_{ij} = S_{ij} + W_{ij}$ into a symmetric (S_{ij}) and an anti-symmetric part (W_{ij}) defined as

$$S_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}), \qquad (5)$$
$$W_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

The three invariants of the tensor A_{ij} can be obtained from the characteristic function of a second order tensor and it turns out that one of the latter (its trace) must vanish for incompressible flows. Then the different types of stagnation points are characterized by the two remaining invariants Q and R and the sign of the determinant $D = Q^3 + 27/4R^2$ yielding the four different possible types of stagnation points. For details on the characterization see [17].

CHARACTERIZATION OF THE ISOSURFACE NEAR STAGNATION POINTS: While the isosurface of a turbulent scalar field can be expected to globally posses a fractal dimension in the high Reynolds number [19], it is locally diffusion controlled and smooth [20]. However, it remains to clarify the local topology of the surface in the vicinity of a stagnation point.

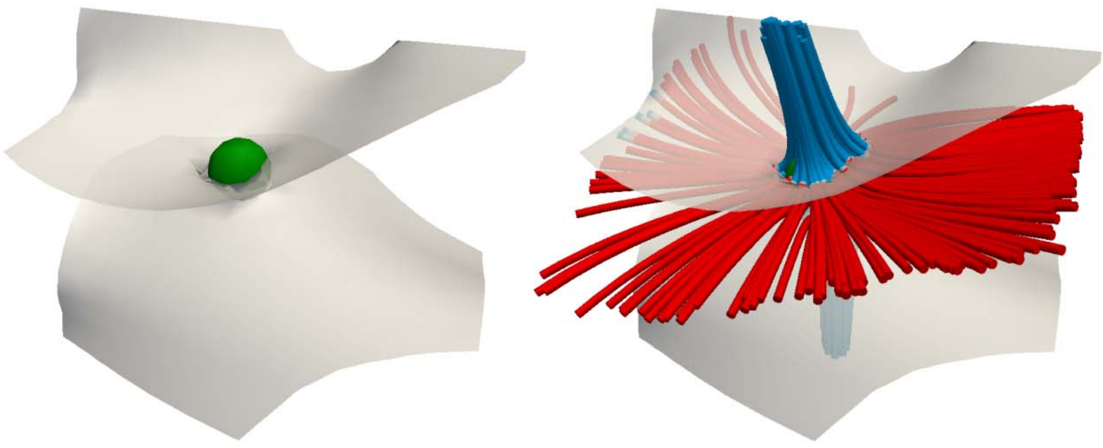


Figure 6: (a) Local topology of the $u_s = 0$ isosurface in the vicinity of a vortex dominated stagnation point (green dot). (b) Streamlines passing through the stagnation point color coded with local sign of local acceleration (red: positive, blue: negative) along the streamline.

To this end we expand the u_s-field based on eq. (4) and find

$$u_s = \frac{\partial u}{\partial s} = t_j \frac{\partial u}{\partial x_j} = t_i t_j \frac{\partial u_i}{\partial x_j} = 0 = u_i u_j \frac{\partial u_i}{\partial x_j}, \tag{6}$$

where the latter identity is only valid on the isosurface itself and does not allow the treatment of u_s as a field any more. Replacing the velocity field in eq. 6 by its expansion (eq. 4) we obtain

$$u_s \approx A_{ik} A_{jl} A_{ij} x_k x_l = B_{kl} x_k x_l = 0. \tag{7}$$

Due to the symmetry of the last equality (B_{kl} itself is not symmetric) we introduce the symmetric tensor

$$\tilde{B}_{kl} = \frac{1}{2}(B_{kl} + B_{lk}) = \frac{1}{2} A_{ij}(A_{ik} A_{jl} + A_{il} A_{jk}), \tag{8}$$

to finally obtain

$$\tilde{B}_{kl} x_k x_l = 0. \tag{9}$$

$\tilde{B}_{kl}$ can then be diagonalized based on its eigenbasis to show that in the vicinity of a stagnation point the solution of eq. 9 yields a locally quadric surface. Due to incompressibility at least one (and no more than two) of the real eigenvalues of $\tilde{B}_{kl}$ must be negative so that only a degenerated quadric surface can be solution to eq. 9, namely that of a cone

$$\left(\frac{\tilde{x}_1}{a_1}\right)^2 + \left(\frac{\tilde{x}_2}{a_2}\right)^2 - \left(\frac{\tilde{x}_3}{a_3}\right)^2 = 0, \tag{10}$$

where we have denoted with $a_1 \dots a_3$ the principle axis of the quadric whose lengths are determined by the absolute value of the eigenvalues of the tensor $\widetilde{B}_{kl}$. We thus conclude from the above analysis that at a stagnation point the isosurface containing all local extreme points of the turbulent kinetic energy field posses a singular point at which two folds of the surface come infinitesimally close to each other.

Figure 6(a) shows the $u_s = 0$ isosurface in the vicinity of a stagnation point in the fluctuating turbulent velocity field. Locally, the surface is clearly a degenerated cone-type quadric as defined by eq. 10 where in the stagnation point two folds of the isosurface "touch" each other. Figure 6(b) shows the local flow structure in terms of streamlines passing through the stagnation point. The color coding of the streamlines indicates the local flow acceleration along the streamline in red ($u_s > 0$) and a local decceleration in blue ($u_s < 0$). As the isosurface devides space into two regions, where in one region $u_s > 0$, which corresponds to the volume above and below the stagnation point, while in the other one $u_s > 0$ which corresponds to the volume between the two folds of the isosurface. Following the terminology by Chong [18] this specific type of stagnation point is an unstable-node/saddle/saddle (corresponding to the flow behavior in the three eigenplanes) characterized by a negative discriminant D which also yields a negative value of the tensor invariant Q and a positive value of the tensor invariant R. Such stagnation points characterize a locally strain-rate dominated region.

Figure 7(b) shows again the $u_s = 0$ isosurface in the vicinity of a stagnation point which this time corresponds to a stable-focus/stretching characterized by positive values of D and R and a negative value of R. Such a stagnation point lies in a flow region which is locally dominated by vortex structures. Again, the isosurface is locally a degenerated cone-type quadric.

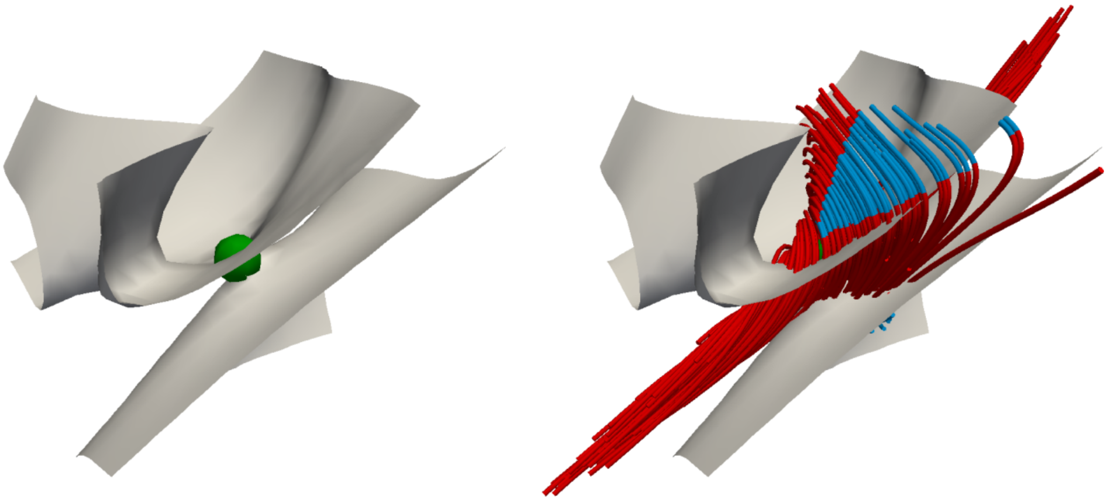


Figure 7: (a) Local topology of the $u_s = 0$ isosurface in the vicinity of a vortex dominated stagnation point (green dot). (b) Streamlines passing through the stagnation point color coded with local sign of local acceleration (red: positive, blue: negative) along the streamline.

CONCLUSION: Different geometrical aspects of turbulent flows have been analyzed in the course of this work, namely dissipation elements, streamline segements and stagnation points. An isosurface, namely $du/ds = 0$ has turned out to be a unifying object of the above geometrical features in the sense that in this surface not only streamline segments, but also dissipation elements based on the instantaneous turbulent kinetic energy field begin and end. When passing through this surface, all streamlines are locally parallel to each other as based on the continuity equation the Gaussian curvature of streamlines vanishes in the isosurface. It could also be shown that the surface comprises all local extrema of the turbulent kinetic energy at any instant in time. Among the latter are also stagnation points, which form a sub-group of the local minima of the turbulent kinetic energy field, namely all absolute minima with $\nabla u = \nabla k = 0$. These are critical points of the velocity field where all three components vanish simultaneously. The surface could be shown to be of a special type in the vicinity of stagnation points where independent of the type of stagnation point two folds of the surface touch and the local expansion yields a degenerated cone-type surface.

ACKNOWLEDGEMENT: This work was funded by the NRW-Research School "BrenaRo" and the cluster of Excellence "Tailor-Made Fuels from Biomass", which is funded by the Excellence Initiative of the German federal state governments to promote science and research at German universities.

References

1. Kolmogorov A.N. *The local structure of turbulence in an incompressible viscous fluid for very large Reynolds numbers.* Dokl. Akad. Nauk SSSR 1941 **30**, 301–305
2. Frisch, U. *Turbulence*. Cambridge University Press, Cambridge, 1995.
3. Corrsin S. *Random geometric problems suggested by turbulence.* Statistical Models and Turbulence. Lecture Notes in Physics. 1971 **12**, edited by M. Rosenblatt and C. van Atta 300–316
4. She Z.E. et al. *Intermittent vortex structures in homogeneous isotropic turbulence.* Nature (London) 1990 **344**, 226–228
5. Kaneda Y. & Ishihara T. *High-resolution direct numerical simulation of turbulence.* J. Turbul. 2006 **7**, 1–17
6. Wang L. and Peters N. *The length scale distribution function of the distance between extremal points in passive scalar turbulence.* J. Fluid Mech. 2006 **554**, 457–475
7. Wang L. & Peters N. *Length scale distribution functions and conditional means for various fields in turbulence.* J. Fluid Mech. 2008 **608**, 113–138
8. Gibson C.H. *Fine structure of scalar fields mixed by turbulence. I. Zero-gradient points and minimal gradient surfaces.* Phys. Fluids 1968 **11**, 2305–2315
9. Schaefer P. *Testing of Model Equations for the Mean Dissipation using Kolmogorov Flows.* Flow Turbulence Combust (2010) **85**, 225–243
10. Wang L. *On properties of fluid turbulence along streamlines.* J. Fluid Mech. 2010 **648**, 183–203
11. Rao P. *Geometry of streamlines in fluid flow theory*. Def. Sci. J. 1978 **28**, 175–178
12. Braun W. et al. *Geometry of particle paths in turbulent flows.* J. Turbul. 2006 **7**, 1–10
13. Scagliarini. A. *Geometric properties of particle trajectories in turbulent flow.* Journal of Turbulence 2011 **12**
14. Schaefer P. et al. *Curvature statistics of streamlines in various turbulent flows.* submitted to J. of Turb. 2012
15. Goto S. *Acceleration statistics as measures of statistical persistence of streamlines in isotropic turbulence.* Phys. Rev. 2005 **E 71**, 015301(R)
16. Goto S. and Vassilicos J.C. *Particle pair diffusion and persistent streamline topology in two-dimensional turbulence.* New J. Phys. 2004 **6**, 65
17. Soria J. & Cantwell B. *Topological visualisation of fucal structures in free shear flows.* Applied Scienti_c Research 1994 **53**, 375 – 386.
18. Chong M. et al. *A general classification of three-dimensional flow fields.* Phys. Fluids A 1990 **2**(5), 765 – 777
19. Mandelbrot B.B. *On the geometry of homogeneous turbulence, with stress on the fractal dimension of the iso-surfaces of scalar.* J. Fluid Mech, 1975 **72**, 401 – 416.
20. Schaefer P. et al. *The length distribution of streamline segments in homogeneous isotropic decaying turbulence* Phys. Fluids 2012 **24**, 045104